

# On Pesin's entropy formula for dominated splittings without mixed behavior

Dawei Yang

Yongluo Cao\*

## Abstract

For  $C^1$  diffeomorphisms, we prove that the Pesin's entropy formula holds for some invariant measure supported on any topological attractor that admits a dominated splitting without mixed behavior. We also prove Shub's entropy conjecture for diffeomorphisms having such kind of splittings.

## 1 Introduction

Pesin's entropy formula characterize the relationship between the metric entropy and Lyapunov exponents: the metric entropy is the integration of the sum of positive Lyapunov exponents. Sometimes, a measure that satisfies the Pesin's entropy formula is called an *SRB* measure when there is at least one positive Lyapunov exponent. We would like to know the existence of measures that satisfy the entropy formula for a given system. Lots of results were got for  $C^2$  maps. Since the absence of distortion bounds, we lose some method to get SRB measures for  $C^1$  maps. However, there are results for  $C^1$  maps. See [4, 6, 14, 19, 20] for instance.

In this paper, we consider a topological attractor which admits a dominated splitting without mixed behavior. We show the existence of measures satisfying Pesin's entropy formula for this kind of systems. Such a splitting is satisfied in some natural setting, for instance, if a non-periodic transitive set of a surface diffeomorphism has a non-trivial dominated splitting, then this dominated splitting has no mixed behavior.

Let  $f$  be a diffeomorphism on a manifold  $M$  whose dimension is  $d$ . For a compact invariant set  $\Lambda$ , one says that  $\Lambda$  admits a *dominated splitting* if there is a continuous invariant splitting  $T_\Lambda M = E \oplus F$ , and constants  $C > 0$  and  $\lambda \in (0, 1)$  such that for any  $x \in \Lambda$ , and  $n \in \mathbb{N}$ , any  $u \in E(x) \setminus \{0\}$  and any  $v \in F(x) \setminus \{0\}$ , we have

$$\frac{\|Df^n(u)\|}{\|u\|} \leq C\lambda^n \frac{\|Df^n(v)\|}{\|v\|}.$$

We say a dominated splitting  $T_\Lambda M = E \oplus F$  has *no mixed behavior* if for any measure  $\mu$  supported on  $\Lambda$ , every Lyapunov exponent of  $\mu$  along  $E$  is non-positive and every Lyapunov exponent of  $\mu$  along  $F$  is non-negative. Equivalently, we have that

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \|Df^n|_{E(x)}\| \leq 0, \quad \liminf_{n \rightarrow \infty} \frac{1}{n} \log \|Df^{-n}|_{F(x)}\| \leq 0, \quad \forall x \in \Lambda.$$

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**Theorem A.** *For a  $C^1$  diffeomorphism  $f$ , if an attractor  $\Lambda$  admits a dominated splitting  $T_\Lambda M = E \oplus F$  without mixed behavior, then there is a measure  $\mu$  supported on  $\Lambda$  satisfying Pesin's entropy formula.*

In a recent paper by Liu and Lu [10], for a  $C^2$  map, they got measures satisfying Pesin's entropy formula for a topological attractor which admits a partially hyperbolic splitting without mixed behavior. Cowieson and Young proved the existence of SRB measures [7, Corollary 1] if  $\Lambda$  is an attractor of a  $C^\infty$  diffeomorphism  $f$  and  $\Lambda$  admits a dominated splitting  $T_\Lambda M = E \oplus F$  without mixed behavior and  $\limsup_{n \rightarrow \infty} (1/n) \log \|Df^n|_{F(x)}\| > 0$  for any point  $x \in \Lambda$ .

With some additional effort from the proof of Theorem A, we can know that the topological entropy varies upper semi continuous w.r.t. the diffeomorphisms. Thus, by a usual argument we can know the entropy conjecture is also true for dominated splittings without mixed behavior.

The diffeomorphism  $f$  induces naturally a map  $f_{*,k} : H_k(M, R) \rightarrow H_k(M, R)$  for any  $0 \leq k \leq d$ , where  $H_k(M, R)$  is the  $k$ -th homology group of  $M$ . Shub conjectured in [17] that for every  $C^1$  diffeomorphism  $f$ ,

$$\max_{0 \leq i \leq d} \text{sp}(f_{*,i}) \leq h_{\text{top}}(f),$$

where  $\text{sp}(A)$  is the spectral radius of a linear map  $A$ .

**Theorem B.** *For a  $C^1$  diffeomorphism  $f$ , if  $M$  admits a dominated splitting without mixed behavior, then the entropy conjecture is true, i.e.,*

$$\max_{0 \leq i \leq d} \text{sp}(f_{*,i}) \leq h_{\text{top}}(f).$$

Shub's entropy conjecture is still open. However, there are lots of interesting results on that. We give a partial list:

- [22] proved that Shub's conjecture holds for  $C^\infty$  maps.
- [15, 18] proved the conjecture for Anosov systems and general Axiom A diffeomorphisms.
- [12] proved the conjecture for the three-dimensional case.
- [16] proved the conjecture for partially hyperbolic systems with one-dimensional center bundle.
- [11] proved the conjecture for diffeomorphisms that away from ones with a homoclinic tangency.
- [10] proved the conjecture for diffeomorphisms admits a partially hyperbolic splitting without mixed behavior.

We notice that the assumption of Theorem B is not contained in any result listed above.

We will also consider the properties of asymptotically entropy expansive and principal symbolic extension in Section 3.5.

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## 2 Definitions and Properties of entropies

In this section, we give the definitions and properties of metric entropy, local entropy and topological entropy.

### 2.1 Metric entropies

Let  $\mu$  be a probability measure. For a finite measurable partition  $\mathcal{B} = \{B_1, B_2, \dots, B_k\}$ , we define

$$H_\mu(\mathcal{B}) = \sum_{i=1}^k -\mu(B_i) \log \mu(B_i),$$

and

$$\bigvee_{i=0}^{n-1} f^{-i}(\mathcal{B}) = \{C : C = \bigcap_{i=0}^{n-1} f^{-i}(B_{i_j})\}.$$

If  $\mu$  is an invariant probability measure of a map  $f$ , the metric entropy of  $\mu$  w.r.t. a partition  $\mathcal{B}$  is

$$h_\mu(f, \mathcal{B}) = \lim_{n \rightarrow \infty} \frac{1}{n} H_\mu\left(\bigvee_{i=0}^{n-1} f^{-i}(\mathcal{B})\right),$$

and the metric entropy of  $\mu$  is

$$h_\mu(f) = \sup_{\mathcal{B}: \text{partition}} h_\mu(f, \mathcal{B}).$$

**Definition 2.1.** Given a finite partition  $\mathcal{B} = \{B_1, B_2, \dots, B_n\}$ , the norm of the partition is  $\max_{1 \leq i \leq n} \text{Diam}(B_i)$ . The norm of  $\mathcal{B}$  is denoted by  $\|\mathcal{B}\|$ .

Given a measure  $\mu$ , a partition  $\mathcal{B}$  is called regular if  $\mu(\partial B) = 0$  for any  $B \in \mathcal{B}$ ; it is called  $\alpha$ -regular if  $\|\mathcal{B}\| < \alpha$  and it is regular.

By the definition, we have the following lemma:

**Lemma 2.2.** Given a regular partition  $\mathcal{B}$  of a measure  $\mu$  of a diffeomorphism  $f$ , and given  $n \in \mathbb{N}$ , for any  $\varepsilon > 0$ , there is  $\delta > 0$  such that for any  $g$  which is  $\delta$ - $C^1$ -close to  $f$ , for any invariant measure  $\nu$  of  $g$  which is  $\delta$ -close to  $\mu$  in the weak-\* topology, then

$$\left| \frac{1}{n} H_\mu\left(\bigvee_{i=0}^{n-1} f^{-i}(\mathcal{B})\right) - \frac{1}{n} H_\nu\left(\bigvee_{i=0}^{n-1} g^{-i}(\mathcal{B})\right) \right| < \varepsilon.$$

The following fundamental results are from [21, Section 8.2]:

**Lemma 2.3.** Let  $\mu_1, \mu_2, \dots, \mu_n$  be probability measures and  $s_1, s_2, \dots, s_n$  be non-negative numbers such that  $\sum s_i = 1$ . For any partition  $\mathcal{B}$ , we have

$$\sum_{i=1}^n s_i H_{\mu_i}(\mathcal{B}) \leq H_{\sum_{i=1}^n s_i \mu_i}(\mathcal{B}).$$

## 2.2 Local entropy

We need to define the Bowen balls or dynamical balls in the entropy theory. Given a point  $x$  and  $\alpha > 0$ ,

- the closed ball of radius  $\alpha$  at  $x$ :  $B(x, \alpha) = \{y \in M : d(x, y) \leq \alpha\}$ ;
- $n$ -th Bowen ball for  $f$ :  $B_n(x, \alpha, f) = \bigcap_{0 \leq i \leq n-1} f^{-i}(B(f^i(x), \alpha))$ ; for simplicity, we denote  $B_n(x, \alpha) = B_n(x, \alpha, f)$  if there is no confusion;
- bi- $n$ -th Bowen ball:  $B_{\pm n}(x, \alpha) = \bigcap_{-n+1 \leq i \leq n-1} f^{-i}(B(f^i(x), \alpha))$ ;
- infinite Bowen ball for  $f$ :  $B_\infty(x, \alpha) = B_{+\infty}(x, \alpha) = \bigcap_{n \in \mathbb{N}} f^{-n}(B(f^n(x), \alpha))$ ; for simplicity, we denote  $B_\infty(x, \alpha) = B_\infty(x, \alpha, f)$  if there is no confusion;
- bi-infinite Bowen ball:  $B_{\pm\infty}(x, \alpha) = \bigcap_{n \in \mathbb{Z}} f^{-n}(B(f^n(x), \alpha))$

**Definition 2.4** (Local entropy). *For a compact set  $\Gamma$  (not necessarily invariant), for  $n \in \mathbb{N}$  and  $\delta$ , a finite set  $P \subset \Gamma$  is called an  $(n, \delta)$ -spanning set for  $f$  (or  $(n, \delta, f)$ -spanning set) if  $P \cap B_n(x, \delta)$  is not empty for any  $x \in \Gamma$ . The minimal cardinality of all  $(n, \delta)$ -spanning set is denoted by  $r_n(\Gamma, \delta)$ .*

*Then one can define the entropy of  $\Gamma$  by*

$$h(f, \Gamma) = h(f|_\Gamma) = \lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log r_n(\Gamma, \delta).$$

*When  $\Gamma$  is a compact invariant set, we also call  $h(f|_\Gamma)$  the topological entropy of  $f$  on  $\Gamma$ . Sometimes, one denotes it by  $h_{\text{top}}(f|_\Gamma)$ .*

*We then define the local entropy of the scale  $\alpha$  for a compact set  $\Gamma$  by*

$$h_\alpha(f|_\Gamma) = \sup_{x \in \Gamma} h(f, B_\infty(x, \alpha)).$$

One has the following lemma for spanning sets from Bowen [1, Lemma 2.1].

**Lemma 2.5.** *Assume that  $\Gamma$  is a compact set and  $\varepsilon > 0$ . Let  $0 = t_0 < t_1 < t_2 < \dots < t_r = n$  be integers. If  $P_i$  is a  $(t_{i+1} - t_i, \varepsilon)$ -spanning set of  $f^{t_i}(\Gamma)$  for any  $0 \leq i \leq r-1$ , then*

$$r_n(\Gamma, 2\varepsilon) \leq \prod_{i=0}^{r-1} \#P_i.$$

By using the definition, we have

**Lemma 2.6.** *Given any  $\alpha > 0$ , for any  $x \in M$  and any  $m \in \mathbb{N}$ , we have*

$$h(f, B_{\pm\infty}(x, \alpha)) = h(f, B_{\pm\infty}(f^m(x), \alpha)).$$

*Proof.* For any  $\varepsilon > 0$ , let us fix an  $\varepsilon/4$ -dense set in  $M$  whose cardinality is  $N_\varepsilon$ . Thus for any compact set  $\Gamma$ , there is a  $(1, \varepsilon)$ -spanning set whose cardinality is at most  $N_\varepsilon$ . For any  $n \in \mathbb{N}$ , by Lemma 2.5, we have

$$r_{m+n}(B_{\pm\infty}(x, \alpha), 2\varepsilon) \leq N_\varepsilon^m r_n(B_{\pm\infty}(f^m(x), \alpha), \varepsilon).$$

On the other hand, for any  $\varepsilon > 0$  and any  $n \in \mathbb{N}$ , if  $P_{m+n}$  is an  $(n+m, \varepsilon)$ -spanning set for  $f$  of  $B_{\pm\infty}(x, \alpha)$  satisfying  $\#P_{m+n} = r_{m+n}(B_{\pm\infty}(x, \alpha), \varepsilon)$ , then  $f^m(P_{m+n})$  is an  $(n, \varepsilon)$ -spanning set for  $f$  of  $B_\infty(f^m(x), \alpha)$ . Hence, we have

$$r_{m+n}(B_{\pm\infty}(x, \alpha), \varepsilon) = \#f^m(P_{m+n}) \geq r_n(B_{\pm\infty}(f^m(x), \alpha), \varepsilon).$$

By taking the limits, one can get the conclusion.  $\square$

## 2.3 Local entropies for $f$ and $f^{-1}$

In this subsection, we need to prove the following proposition. We borrow some ideas from [11, Proposition 2.5].

**Proposition 2.7.** *For any ergodic measure  $\mu$ , there is a full  $\mu$ -measure set  $R$  such that*

$$\sup_{x \in R} h(f, B_{\pm\infty}(x, \alpha)) = \sup_{x \in R} h(f^{-1}, B_{\pm\infty}(x, \alpha)).$$

*Proof.* In fact, since  $\mu$  is ergodic, one can take

$$R = \{x : \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x)} \rightarrow \mu, \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^{-i}(x)} \rightarrow \mu\}.$$

In this proposition, the situation for  $f$  and  $f^{-1}$  is symmetric. Without loss of generality, one can assume that there is a point  $x_0 \in R$  such that

$$h(f, B_{\pm\infty}(x_0, \alpha)) > \sup_{x \in R} h(f^{-1}, B_{\pm\infty}(x, \alpha)).$$

Thus, one can find two numbers  $a_1 > a_2$  such that

$$h(f, B_{\pm\infty}(x_0, \alpha)) > a_1 > a_2 > \sup_{x \in R} h(f^{-1}, B_{\pm\infty}(x, \alpha)).$$

Recall the definition of the local entropy, there is  $\varepsilon_0 > 0$  small such that for any  $\varepsilon \in (0, \varepsilon_0)$ , we have

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log r_n(B_{\pm\infty}(x_0, \alpha), \varepsilon) > a_1.$$

In other words, there is a sequence of integers  $\{n_i\}$  such that

$$r_{n_i}(B_{\pm\infty}(x_0, \alpha), \varepsilon) > e^{a_1 n_i}.$$

For this  $\varepsilon_0 > 0$ , we choose a finite set  $P_0$  that  $\varepsilon_0/8$ -dense in  $M$ . Thus, for any compact subset  $\Gamma$  of  $M$ , there is a  $\varepsilon_0/2$ -dense set in  $\Gamma$ , whose cardinality is at most  $\#P_0$ .

Take

$$\mu_{n_i} = \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x_0)}.$$

Since  $\mu$  is ergodic, we have that  $\mu_{n_i} \rightarrow \mu$  as  $i \rightarrow \infty$ .

For  $\mu$ , for each  $n \in \mathbb{N}$ , one can find  $\varepsilon_n \ll \varepsilon_0$  such that if we define the set  $R_n$  as

$$R_n = \{x \in R : r_m(B_{\pm\infty}(x, \alpha), \varepsilon_n/4, f^{-1}) < e^{a_2 m}, \quad \forall m \geq n\},$$

then we have

- $\mu(\cup_{n \in \mathbb{N}} R_n) = 1$ .
- $\{R_n\}$  is an increasing sequence of measurable sets.

Then one can choose an increasing sequence of *compact* sets  $\{\Lambda_n\}$  such that

- $\Lambda_n \subset R_n$  for each  $n \in \mathbb{N}$ .
- $\mu(\cup_{n \in \mathbb{N}} \Lambda_n) = 1$ .

Now we fix some  $n$  that is probably large enough. For any  $x \in \Lambda_n$ , let  $P_n(x)$  be an  $(n, \varepsilon_n/4, f^{-1})$ -spanning set of  $B_{\pm\infty}(x, \alpha)$  such that  $\#P_n(x) < e^{a_2 n}$ . Then

$$U_n(x) = \bigcup_{z \in P_n(x)} B_n(z, \varepsilon_n/2, f^{-1})$$

is a neighborhood of  $B_{\pm\infty}(x, \alpha)$ .

We have the following observations.

- $f^{-n}P_n(x)$  is a  $(n, \varepsilon_n/4)$ -spanning set of  $B_{\pm\infty}(f^{-n}(x), \alpha)$  for  $f$  for any  $x \in R_n$ . It is clear that  $\#f^{-n}P_n(x) < e^{a_2 n}$ .
- $\mu(f^{-n}(\Lambda_n)) = \mu(\Lambda_n)$

Now we choose a smaller neighborhood  $V_n(x) \subset U_n(x)$  of  $x$  and an integer  $N_n(x)$  such that for any  $y \in V_n(x)$ , we have

$$B_{\pm N_n(x)}(y, \alpha) \subset U_n(x).$$

By the definition of  $U_n(x)$ , we have that for any  $y \in V_n(x)$ ,  $B_{\pm N_n(x)}(y, \alpha)$  is  $(n, \varepsilon_n/2, f^{-1})$ -spanned by  $P_n(x)$ .

As a corollary, we have that

$$\{V_n(x)\}_{x \in \Lambda_n}$$

is an open covering of  $\Lambda_n$ . Thus, there are finitely points  $\{x_1, x_2, \dots, x_k\} \subset \Lambda_n$  such that

$$W_n = \bigcup_{1 \leq j \leq k} V_n(x_j) \supset \Lambda_n.$$

Consequently,

$$\lim_{n \rightarrow \infty} \mu(W_n) = 1.$$

Take

$$H(n) = \max\{n, N_n(x_1), \dots, N_n(x_k)\}.$$

We have that

$$\lim_{i \rightarrow \infty} \mu_{n_i}(f^{-n}(W_n)) = \lim_{i \rightarrow \infty} \mu_{n_i}(f^{-n}(W_n)) \geq \mu(f^{-n}(W_n)) = \mu(W_n) \geq \mu(\Lambda_n).$$

Now we consider the positive iteration of  $x_0$ . For  $n_i$  large, we find a sequence of times  $0 = \iota_0 < \iota_1 < \dots < \iota_L = n_i$  by the following way inductively:

- if  $f^{\iota_j} \in f^{-n}(W_n)$  and  $\iota_j \in [H(n), n_i - H(n)]$ , then  $\iota_{j+1} = \iota_j + n$ ;
- otherwise, one takes  $\iota_{j+1} = \iota_j + 1$ .

Let

$$A_n = \{\iota_j : f^{\iota_j}(x_0) \in f^{-n}(W_n), \iota_j \in [H(n), n_i - H(n)]\}, \quad B_n = \{\iota_i\}_{0 \leq j \leq L} \setminus A_n.$$

We have the following properties:

- if  $\iota_j \in A_n$ , then there is some  $k_j \in \{1, 2, \dots, k\}$  such that  $f^{\iota_j+n}(x_0) \in V(x_{k_j})$ , and

$$f^{\iota_j+n}(B_{\pm n_i}(x_0, \alpha)) \subset B_{\pm H(n)}(f^{\iota_j+n}(x_0), \alpha) \subset U_n(x_{k_j}).$$

This implies that  $f^{\iota_j}(B_{\pm n_i}(x_0, \alpha))$  is  $(n, \varepsilon_n/4, f)$ -spanned by  $f^{-n}P_n(x_{k_j})$

By Lemma 2.5, we have

$$r_{n_i}(B_{\pm n_i}(x_0, \alpha), \varepsilon_0/2) \leq \left( \prod_{\iota_j \in A_n} \#f^{-n}P_n(x_{k_j}) \right) \cdot (\#P_0)^{\#B_n} \leq e^{a_2 n \#A_n} \cdot (\#P_0)^{\#B_n},$$

By definitions, we have

- $n \#A_n \leq n_i$ ,
- $\#B_n \leq \#\{0 \leq j < n_i : f^j(x_0) \notin f^{-n}(W_n)\} + 2H(n) \leq (1 - \mu_{n_i}(W_n))n_i + 2H(n)$ .

This implies that

$$r_{n_i}(B_{\pm n_i}(x_0, \alpha), \varepsilon_0/2) \leq e^{a_2 n_i} \cdot (\#P_0)^{(1 - \mu_{n_i}(W_n))n_i + 2H(n)}.$$

Thus,

$$\frac{1}{n_i} \log r_{n_i}(B_{\pm n_i}(x_0, \alpha), \varepsilon_0/2) \leq a_2 + [(1 - \mu_{n_i}(W_n)) + 2\frac{H(n)}{n_i}] \log \#P_0.$$

For fixed  $n$ , we have that  $H(n)$  is much smaller than  $n_i$ . By taking  $n_i \rightarrow \infty$ , we have

$$\limsup_{n_i \rightarrow \infty} \frac{1}{n_i} \log r_{n_i}(B_{\pm \infty}(x_0, \alpha), \varepsilon_0/2) \leq \limsup_{n_i \rightarrow \infty} \frac{1}{n_i} \log r_{n_i}(B_{\pm n_i}(x_0, \alpha), \varepsilon_0/2) \leq a_2 + (1 - \mu(W_n)) \log \#P_0.$$

Then by asking  $n \rightarrow \infty$ ,

$$\limsup_{n_i \rightarrow \infty} \frac{1}{n_i} \log r_{n_i}(B_{\pm \infty}(x_0, \alpha), \varepsilon_0/2) \leq a_2 + \lim_{n \rightarrow \infty} (1 - \mu(W_n)) \log \#P_0 = a_2 < a_1.$$

We get a contradiction. The proof is complete. □

In fact, we have the following more accurate characterization.

**Proposition 2.8.** *For any ergodic measure  $\mu$ , there is a constant  $H$  such that for  $\mu$ -a.e.  $x$ , we have*

$$H = h(f, B_{\pm \infty}(x, \alpha)) = h(f^{-1}, B_{\pm \infty}(x, \alpha)).$$

For proving Proposition 2.8, we need to adapt the definition of the entropy. For a compact invariant set  $\Gamma$ , given  $n \in \mathbb{N}$  and  $\varepsilon > 0$ , a subset  $P$  of  $\Gamma$  is called an  $(n, \varepsilon)$ -separated set of  $\Gamma$  if for any  $x, y \in P$ ,  $d_n(x, y) > \varepsilon$ . Denote by  $s_n(\Gamma, \varepsilon)$  the largest cardinality for any  $(n, \varepsilon)$ -subset of  $\Gamma$ .

By summarizing [21, Chapter 7.2], we have

- $r_n(\Gamma, \varepsilon) \leq s_n(\Gamma, \varepsilon) \leq r_n(\Gamma, \varepsilon/2)$  for any  $n$  and any  $\varepsilon$ .
- $h(f, \Gamma) = \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} (1/n) \log s_n(\Gamma, \varepsilon) = \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} (1/n) \log r_n(\Gamma, \varepsilon)$ .

We need to modify the definition of  $s_n$  to  $\bar{s}_n$  by the following way: for a compact invariant set  $\Gamma$ , given  $n \in \mathbb{N}$  and  $\varepsilon > 0$ , a subset  $P$  of  $\Gamma$  is called an *closed*  $(n, \varepsilon)$ -separated set of  $\Gamma$  if for any  $x, y \in P$ ,  $d_n(x, y) \geq \varepsilon$ . Denote by  $\bar{s}_n(\Gamma, \varepsilon)$  the largest cardinality for any closed  $(n, \varepsilon)$ -separated set of  $\Gamma$ . By using the definitions, we have the following properties:

- Given  $\varepsilon > 0$  and  $n \in \mathbb{N}$ , we have

$$s_n(\Gamma, \varepsilon) \leq \bar{s}_n(\Gamma, \varepsilon) \leq s_n(\Gamma, \varepsilon/2).$$

- $h(f, \Gamma) = \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} (1/n) \log \bar{s}_n(\Gamma, \varepsilon)$ .

Now we can give the proof of Proposition 2.8.

*Proof of Proposition 2.8.* For fixed  $\alpha > 0$ , we define the local entropy function

$$H(x) = h(f, B_{\pm\infty}(x, \alpha)).$$

We need to verify that  $H(x)$  is measurable. After that, by Lemma 2.6, we have  $H(x) = H(f(x))$ , and then by the ergodicity of  $\mu$ , we have that  $H(x)$  is constant for  $\mu$ -a.e.  $x$ . By the same reason, we have that  $h(f^{-1}, B_{\pm\infty}(x, \alpha))$  is also a constant for  $\mu$ -a.e.  $x$ . Then by Proposition 2.7, one can conclude this proposition.

To verify  $H$  is a  $\mu$ -measurable, it is enough to check that given  $\varepsilon > 0$ , the set

$$L_a = \{x : \limsup_{n \rightarrow \infty} \frac{1}{n} \log \bar{s}_n(B_{\pm\infty}(x, \alpha), \varepsilon) > a\}$$

is  $\mu$ -measurable for any  $a > 0$ . We have that

$$L_a = \bigcup_{k \in \mathbb{N}} \bigcap_{n \in \mathbb{N}} \bigcup_{m \geq n} \{x : \bar{s}_m(B_{\pm\infty}(x, \alpha), \varepsilon) \geq e^{(a+1/k)m}\}.$$

Thus it is enough to show that the set

$$L_{a,m} = \{x : \bar{s}_m(B_{\pm\infty}(x, \alpha), \varepsilon) \geq e^{am}\}$$

is  $\mu$ -measurable for any  $a > 0$  and  $m \in \mathbb{N}$ . This can be deduced the fact that  $L_{a,m}$  is closed by the upper semi continuity of bi-infinity Bowen balls.

In fact, assume that there is a sequence  $\{x_n\} \subset L_{a,m}$  such that  $\lim_{n \rightarrow \infty} x_n = x$ , we need to show that  $x \in L_{a,m}$ . For each  $x_n$ , there is a closed  $(m, \varepsilon)$ -separated set  $P_n = \{y_n^1, y_n^2, \dots, y_n^{N_n}\}$  contained in  $B_{\pm\infty}(x_n, \alpha)$  whose cardinality is  $N_n = \#P_n \geq [e^{am}]$ .

By taking a subsequence if necessary, one can assume that for any  $1 \leq j \leq [e^{am}]$ ,  $\lim_{n \rightarrow \infty} y_n^j = y^j \in B_{\pm\infty}(x, \alpha)$ . Moreover, we have that for any  $1 \leq i < j \leq [e^{am}]$ ,  $d_m(y^i, y^j) \geq \varepsilon$ . This implies that there is a closed  $(m, \varepsilon)$ -separated set contained in  $B_{\pm\infty}(x, \alpha)$  whose cardinality is at least  $[e^{am}]$ , and hence  $e^{am}$ . Consequently,  $x \in L_{a,m}$ . The proof is complete now.  $\square$



### 3 Upper semi continuity of entropies

In this section, we will mainly prove the upper semi continuity of the metric entropy w.r.t. invariant measures. Actually, we prove the following stronger result.

**Theorem 3.1.** *Assume that a compact invariant set  $\Lambda$  admits a dominated splitting  $T_\Lambda M = E \oplus F$  without mixed behavior. If there is a sequence of diffeomorphisms  $\{f_n\}$  and a sequence of invariant measures  $\mu_n$  such that each  $\mu_n$  is an invariant measure of  $f_n$  and supported on a compact invariant set  $\Lambda_n$  of  $f_n$ , and*

$$\lim_{n \rightarrow \infty} f_n = f, \quad \lim_{n \rightarrow \infty} \mu_n = \mu, \quad \limsup_{n \rightarrow \infty} \Lambda_n \subset \Lambda,$$

then

$$\limsup_{n \rightarrow \infty} h_{\mu_n}(f_n) \leq h_\mu(f).$$

We first give some consequences of Theorem 3.1, and then give its proof.

#### 3.1 Consequences of Theorem 3.1

One says that the entropy function is *upper semi continuous w.r.t. the measures* if for any measure  $\mu$  and any sequence of measures  $\mu_n$  such that  $\lim_{n \rightarrow \infty} \mu_n = \mu$ , then  $\limsup_{n \rightarrow \infty} h_{\mu_n}(f) \leq h_\mu(f)$ .

**Corollary 3.2.** *Assume that a compact invariant set  $\Lambda$  admits a dominated splitting  $T_\Lambda M = E \oplus F$  without mixed behavior. Then the metric entropy is upper semi continuous w.r.t. the measures.*

The corollary can be deduced from Theorem 3.1 directly.

The upper semi continuity of the entropy function can be applied in thermodynamical formalism. For any continuous function  $\varphi$ , the pressure of  $\varphi$  is defined by

$$P(\varphi) = \sup_{\mu \text{ invariant}} \{h_\mu(f) + \int \varphi d\mu\}.$$

A measure  $\mu$  is called an *equilibrium state* of  $\varphi$  if  $P(\varphi) = h_\mu(f) + \int \varphi d\mu$ . By the upper semi continuity, we have the following corollary directly:

**Corollary 3.3.** *Assume that a compact invariant set  $\Lambda$  admits a dominated splitting  $T_\Lambda M = E \oplus F$  without mixed behavior. Then every continuous function of  $\Lambda$  has an equilibrium state on  $\Lambda$ .*

Another corollary is the upper semi continuity of topological entropy w.r.t. the diffeomorphisms.

**Corollary 3.4.** *Assume that a compact invariant set  $\Lambda$  admits a dominated splitting  $T_\Lambda M = E \oplus F$  without mixed behavior. If  $f_n \rightarrow f$  as  $n \rightarrow \infty$  in the  $C^1$  topology and  $\Lambda_{f_n}$  is a compact invariant set of  $f_n$  satisfying  $\limsup_{n \rightarrow \infty} \Lambda_{f_n} \subset \Lambda$ , then*

$$\limsup_{n \rightarrow \infty} h_{top}(f_n|_{\Lambda_{f_n}}) \leq h_{top}(f|_{\Lambda_f}).$$

*Proof.* For each  $n$ , we take an ergodic measure  $\mu_n$  supported on  $\Lambda_{f_n}$  such that

$$h_{\mu_n}(f_n|_{\Lambda_{f_n}}) > h_{top}(f_n|_{\Lambda_{f_n}}) - \frac{1}{n}.$$

By taking a subsequence if necessary, we assume that  $\lim_{n \rightarrow \infty} \mu_n = \mu$  for some invariant measure  $\mu$  of  $f$ . By Theorem 3.1, we have that

$$h_{top}(f|_{\Lambda_f}) \geq h_{\mu}(f|_{\Lambda_f}) \geq \limsup_{n \rightarrow \infty} h_{\mu_n}(f_n|_{\Lambda_{f_n}}) = \limsup_{n \rightarrow \infty} \left( h_{\mu_n}(f_n|_{\Lambda_{f_n}}) + \frac{1}{n} \right) = \limsup_{n \rightarrow \infty} h_{top}(f_n|_{\Lambda_{f_n}}).$$

□

*Proof of Theorem B.* Now we consider a diffeomorphism  $f$  such that  $M$  admits a dominated splitting without mixed behavior. There is a neighborhood  $\mathcal{U}$  of  $f$  such that any  $g \in \mathcal{U}$  is isotropic to  $f$ . Thus we have

$$\max_{0 \leq i \leq d} \text{sp}(f_{*,i}) = \max_{0 \leq i \leq d} \text{sp}(g_{*,i}),$$

For any  $\varepsilon > 0$ , we choose a  $C^\infty$  diffeomorphism  $g \in \mathcal{U}$  such that by applying Yomdin's result [22], we have

$$h_{top}(f) > h_{top}(g) - \varepsilon \geq \max_{0 \leq i \leq d} \text{sp}(g_{*,i}) - \varepsilon = \max_{0 \leq i \leq d} \text{sp}(f_{*,i}) - \varepsilon.$$

Then by the arbitrariness of  $\varepsilon$ , one can complete the proof.

□

### 3.2 Uniformity on dominated splittings without mixed behavior

**Lemma 3.5.** *Assume that  $\Lambda$  admits a dominated splitting without mixed behavior. Then for any  $\beta > 0$ , there is  $N = N(\beta) \in \mathbb{N}$  and a neighborhood  $\mathcal{U}$  of  $f$  such that for any  $g \in \mathcal{U}$  and a neighborhood  $U$  of  $\Lambda$ , for any compact invariant set  $\Lambda_g$  of  $g$  that is contained in  $U$ , we have that  $\Lambda_g$  admits a dominated splitting*

$$T_{\Lambda_g} M = E_g \oplus F_g,$$

and

$$\|Dg^N|_{E_g(x)}\| \leq (1 + \beta)^N, \quad \|Dg^{-N}|_{F_g(x)}\| \leq (1 + \beta)^N.$$

*Proof.* By the main techniques in [5], for  $\beta/2$ , there is  $N > 0$  such that for any  $x \in \Lambda$ , we have

$$\|Df^N|_{E(x)}\| \leq (1 + \beta/2)^N, \quad \|Df^{-N}|_{F(x)}\| \leq (1 + \beta/2)^N.$$

Thus there is a neighborhood  $\mathcal{U}$  of  $f$  such that for any  $g \in \mathcal{U}$ , if a compact invariant set  $\Lambda_g$  of  $g$  is contained in a small neighborhood of  $\Lambda$ , then  $\Lambda_g$  has a dominated splitting  $T_{\Lambda_g} M = E_g \oplus F_g$ . By shrinking  $\mathcal{U}$  and  $U$  if necessary, we have that  $E_g$  and  $F_g$  are close to  $E$  and  $F$ , respectively. Thus for any  $x \in \Lambda_g$ , we have

$$\|Dg^N|_{E_g(x)}\| \leq (1 + \beta)^N, \quad \|Dg^{-N}|_{F_g(x)}\| \leq (1 + \beta)^N.$$

□

### 3.3 The plaque family theorem and Pliss Lemma

For dominated splittings, we have local invariant center stable manifolds and local invariant center unstable manifolds [8, Theorem 5.5].

For  $i \in \mathbb{N}$ , denote by  $D^i(1)$  be the unit ball in  $\mathbb{R}^i$  and  $\text{Emb}(D^i(1), M)$  is the space of  $C^1$  embeddings from  $D^i(1)$  to  $M$ .

**Lemma 3.6.** *Let  $\Lambda$  be a compact invariant set with a dominated splitting  $T_\Lambda M = E \oplus F$ , where  $\dim E = i$ . Then there is a neighborhood  $\mathcal{U}$  of  $f$  and a neighborhood  $U$  of  $\Lambda$  such that for any  $g \in \mathcal{U}$ , for any compact invariant set  $\Lambda_g$  contained in  $U$ , denoting the dominated splitting of  $\Lambda_g$  by  $E_g \oplus F_g$ , then there is a map  $\Theta_g : \Lambda_g \rightarrow \text{Emb}(D^i(1), M)$  such that when one denotes  $W_\varepsilon^{E_g}(x, g) = \Theta_g(x)(D^i(\varepsilon))$ , we have*

- *Invariance:* for any  $\varepsilon > 0$ , there is  $\delta > 0$  such that for any  $x \in \Lambda_g$ , we have  $g(W_\delta^{E_g}(x, g)) \subset W_\varepsilon^{E_g}(g(x), g)$ .
- *Tangency:* for any  $x \in \Lambda_g$ , we have  $T_x W_\varepsilon^{E_g}(x, g) = E_g(x)$ .
- *Continuity:* when  $g_n \rightarrow g$  as  $n \rightarrow \infty$ ,  $x_n \in \Lambda_{g_n}$  such that  $x_n \rightarrow x \in \Lambda$ , then  $W^{E_{g_n}}(x_n, g_n) \rightarrow W^{E_g}(x, g)$ .

We can also get the manifolds  $\{W^F(x, g)\}_{x \in \Lambda_g}$  tangent to  $F_g$ .

**Remark.** We notice that the continuity is not stated in the original version of the plaque family theorem. From the proof of the plaque family theorem, one can know this property.

We have the following version of Pliss lemma [13] that is useful to get uniform estimations in some non-uniform setting. Recall that  $m(A)$  is the mini-norm of a linear isomorphism  $A$ , i.e.,  $m(A) = \inf_{\|v\|=1} \|Av\|$ .

**Lemma 3.7.** *Assume that  $\Lambda$  is a compact invariant set with a dominated splitting  $T_\Lambda M = E \oplus F$ . Given  $N \in \mathbb{N}$  and  $\lambda_1 > \lambda_2 > 1$  such that for any  $x \in \Lambda$ , if*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log m(Df^N|_{F(f^{iN}(x))}) \geq \lambda_1,$$

*then there is a point  $y$  in the positive orbit of  $x$ , we have for any  $n \in \mathbb{N}$*

$$\frac{1}{n} \sum_{i=0}^{n-1} \log m(Df^N|_{F(f^{iN}(y))}) \geq \lambda_2.$$

We have the following estimations on centre-unstable manifolds. The proof is a simple application of the mean value theorem, hence omitted.

**Lemma 3.8.** *Assume that  $\Lambda$  is a compact invariant set with a dominated splitting  $T_\Lambda M = E \oplus F$ . Given  $n \in \mathbb{N}$ , for  $\lambda_1 > \lambda_2 > 1$ , there are  $C = C(\lambda_1, \lambda_2)$  and  $\alpha_0 = \alpha_0(\lambda_1, \lambda_2)$ , for any  $x \in \Lambda$  satisfying*

$$\prod_{i=0}^{n-1} m(Df^N|_{F(f^{iN}(y))}) \geq \lambda_1^n, \quad \forall n \in \mathbb{N}$$

*for any  $y$  and for any  $n \in \mathbb{N}$  such that*

$$f^\ell(y) \in W_{\alpha_0}^F(f^\ell(x)), \quad \forall 0 \leq \ell \leq n,$$

*then we have*

$$d(f^n(x), f^n(y)) \geq C\lambda_2^n d(x, y).$$

### 3.4 The entropy of a plaque

**Lemma 3.9.** *Let  $\Lambda$  be a compact invariant set that admits a dominated splitting  $T_\Lambda M = E \oplus F$  without mixed behavior. For any  $\varepsilon > 0$ , there is a neighborhood  $\mathcal{U}$  of  $f$  and a neighborhood  $U$  of  $\Lambda$  and  $\alpha > 0$  such that for any  $g \in \mathcal{U}$  and for any point  $x \in \Lambda_g \subset U$ , we have*

$$h(g|_{W_\alpha^E(x)}) \leq \varepsilon, \quad h(g^{-1}|_{W_\alpha^F(x)}) \leq \varepsilon.$$

*Proof.* We only prove the case for  $W^E$ . The result for  $W^F$  will be symmetric.

Given  $\varepsilon > 0$ , we take  $\beta > 0$  such that  $(\dim E) \log(1 + 2\beta) < \varepsilon$ . By Lemma 3.5, there is  $N = N(\beta) \in \mathbb{N}$  and a neighborhood  $\mathcal{U}$  of  $f$  and a neighborhood  $U$  of  $\Lambda$  such that for any  $g \in \mathcal{U}$ , for any compact invariant set  $\Lambda_g$  of  $g$  in  $U$ , we have that  $\Lambda_g$  admits a dominated splitting

$$T_{\Lambda_g} M = E_g \oplus F_g,$$

Now we have that for any  $x \in \Lambda_g$ ,

$$\|Dg^N|_{E_g(x)}\| \leq (1 + \beta)^N.$$

Thus one can choose  $\alpha > 0$  small such that for any  $z \in W_\alpha^{E_g}(x)$ , we have

$$\|Dg^N|_{T_z W^{E_g}(x)}\| \leq (1 + 2\beta)^N.$$

Thus, if we take  $C = C(\beta)$  to be

$$C = \max\{(1 + 2\beta)\|Df\|, (1 + 2\beta)^2\|Df^2\|, \dots, (1 + 2\beta)^{N-1}\|Df^{N-1}\|\} + 2.$$

Then we have for any  $z \in W_\alpha^{E_g}(x)$  and any  $n \in \mathbb{N}$ , if  $g^\ell(z) \subset W_\alpha^{E_g}(g^\ell(x))$  for any  $0 \leq \ell \leq n - 1$ , then we have that

$$\|Dg^n|_{T_z W^{E_g}(x)}\| \leq C(1 + 2\beta)^n.$$

Fix  $\delta > 0$ . By using the Mean Value Theorem, for any  $y, z \in W_\alpha^{E_g}(x)$  satisfying  $d(y, z) < \delta$ , when  $f^\ell(y), f^\ell(z) \in W_\alpha^{E_g}(f^\ell x)$  for any  $1 \leq \ell \leq n - 1$ , there is  $\xi_n \in W_\alpha^{E_g}(x)$  such that

$$d(g^n(y), g^n(z)) \leq \|Dg^n|_{T_{\xi_n} W^{E_g}(x)}\| d(y, z) \leq C(1 + 2\beta)^n d(y, z).$$

Thus, the  $n$ -th Bowen ball  $B_n(y, \delta)$  contains a ball of radius  $\delta/C(1 + 2\beta)^n$ . We consider the volume of the ball  $B_n(y, \delta)$ , then we have

$$\text{Volume}(B_n(y, \delta)) \geq \frac{\delta^{\dim E}}{C^{\dim E} (1 + 2\beta)^{n \dim E}}.$$

Thus, there are at most

$$\left\lceil \frac{C^{\dim E} \alpha^{\dim E} (1 + 2\beta)^{n \dim E}}{\delta^{\dim E}} \right\rceil + 1$$

disjoint  $n$ -th Bowen balls contained in  $W_\alpha^{E_g}(x)$ . This implies the entropy is bounded by

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log \frac{C^{\dim E} \alpha^{\dim E} (1 + 2\beta)^{n \dim E}}{\delta^{\dim E}} \leq (\dim E) \log(1 + 2\beta) < \varepsilon.$$

□

### 3.5 Estimation of the local entropy

We need the following lemma for local entropy.

**Lemma 3.10.** *Assume that a compact invariant set  $\Lambda$  admits a dominated splitting  $T_\Lambda M = E \oplus F$  without mixed behavior. Then for any  $\varepsilon > 0$ , there is  $\alpha > 0$  and a neighborhood  $\mathcal{U}$  of  $f$  and a neighborhood  $U$  of  $\Lambda$  such that for any  $g \in \mathcal{U}$  and any compact invariant set  $\Lambda_g \subset U$  of  $g$ , we have*

$$h_\alpha(g|_{\Lambda_g}) \leq \varepsilon.$$

*Proof.* We recall a result from [11, Proposition 2.5]. For proving  $h_\alpha(g) \leq \varepsilon$ , it suffices to prove that for any ergodic invariant measure  $\mu$  supported on  $\Lambda_g$  of  $g$ , for  $\mu$ -a.e.  $x$ , for the bi-infinite Bowen ball, we have that

$$h(g, B_{\pm\infty}(x, \alpha)) \leq \varepsilon.$$

In fact, by Proposition 2.7, it suffices to prove that

- either,  $h(g, B_{\pm\infty}(x, \alpha)) \leq \varepsilon$  for  $\mu$ -a.e.  $x$ ;
- or,  $h(g^{-1}, B_{\pm\infty}(x, \alpha)) \leq \varepsilon$  for  $\mu$ -a.e.  $x$

For the constants of the dominated splitting, we assume that there are  $N \in \mathbb{N}$  and  $\lambda \in (0, 1)$  (independent of  $g$ ) such that for any  $x \in \Lambda_g$ ,  $\|Dg^N|_{E_g(x)}\| \|Dg^{-N}|_{F_g(g^N(x))}\| \leq \lambda$ .

We define the functions

$$\varphi^{E_g}(x) = \log \|Dg^N|_{E_g(x)}\|, \quad \psi^{F_g}(x) = \log m(Dg^N|_{F_g(x)}).$$

$$S_n(\varphi^{E_g}(x)) = \frac{1}{n} \sum_{i=0}^{n-1} \varphi^{E_g}(g^{iN}(x)), \quad S_n(\psi^{F_g}(x)) = \frac{1}{n} \sum_{i=0}^{n-1} \psi^{F_g}(g^{iN}(x)).$$

By Birkhoff's ergodic theorem, the following two limits exist:

$$\lim_{n \rightarrow \infty} S_n(\varphi^{E_g}(x)) = \int \varphi^{E_g}(x) d\mu, \quad \lim_{n \rightarrow \infty} S_n(\psi^{F_g}(x)) = \int \psi^{F_g}(x) d\mu.$$

By domination, at most one of the above quantities is contained in  $(\log \lambda/2, -\log \lambda/2)$  for  $\mu$ -a.e.  $x$ .

Without loss of generality, one assume that  $\lim_{n \rightarrow \infty} S_n(\psi^{F_g}(x))$  is not contained in this interval. Thus, we have that  $\lim_{n \rightarrow \infty} S_n(\psi^{F_g}(x)) \geq -\log \lambda/2$ . In this case, we will prove that for  $\mu$ -a.e.  $x$ ,  $B_{\pm\infty}(x, \alpha) \subset W_\alpha^E(x)$ , and by applying Lemma 3.9, one can conclude.

Notice that when  $\lim_{n \rightarrow \infty} S_n(\psi^{E_g}(x))$  is not in this interval, then one can also prove that for  $\mu$ -a.e.  $x$ ,  $B_{\pm\infty}(x, \alpha) \subset W_\alpha^F(x)$ . Then we need to apply Proposition 2.7 to prove that for  $\mu$ -a.e.  $x$ , we have that  $h(f^{-1}, B_{\pm\infty}(x, \alpha))$  is small.

Take  $C = C(\lambda^{-1/4}, \lambda^{-1/5})$  and  $\alpha_0 = \alpha_0(\lambda^{-1/4}, \lambda^{-1/5})$  as in Lemma 3.8. By reducing  $\alpha_0$  if necessary, one can assume that for any  $w_1, w_2$  in some locally maximal invariant set of some neighborhood of  $\Lambda$ , if  $d(w_1, w_2) < \alpha_0$ , then

$$\lambda^{1/12} \leq \frac{m(Df^N|_{F(w_1)})}{m(Df^N|_{F(w_2)})} \leq \lambda^{-1/12}.$$

The above reduction implies

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \left( \prod_{i=0}^{n-1} m(Df^N|_{F(f^{iN}(x))}) \right) \geq -\frac{\log \lambda}{2}.$$

By Lemma 2.6, it is enough to estimate the entropy at any iterate of  $x$ . By Lemma 3.7, without loss of generality after an iteration, one can assume that

$$\prod_{i=0}^{n-1} m(Df^N|_{F(f^{iN}(x))}) \geq \lambda^{-n/3}, \quad \forall n \in \mathbb{N}.$$

By reducing  $\alpha$  if necessary, since  $y \in B_{\pm\infty}(x, \alpha)$ , we have

$$\prod_{i=0}^{n-1} m(Df^N|_{F(f^{iN}(y))}) \geq \lambda^{-n/4}, \quad \forall n \in \mathbb{N}.$$

If there is  $y \in B_{\pm\infty}(x, \alpha) \setminus W_{\alpha_0}^{E_g}(x)$ , then we consider  $z \in W_{\alpha_0}^{F_g}(y) \cap W_{\alpha_0}^{E_g}(x)$ . There is  $n_0$  such that

- such that  $d(g^{n_0}(y), g^{n_0}(z))$  is almost  $\alpha_0$  by Lemma 3.8. This means that  $n_0$  is related to  $\alpha$ : when  $\alpha$  is small we have  $n_0$  is large.
- $d(g^{n_0}(x), g^{n_0}(z))/d(g^{n_0}(y), g^{n_0}(z))$  is small when  $n_0$  is large by the domination.
- $d(g^{n_0}(x), g^{n_0}(y))$  is bounded by  $\alpha$  since  $y$  is contained in the Bowen ball of  $x$  of size  $\alpha$ .

When  $\alpha \ll \alpha_0$ , we have that  $n_0$  is large. Thus,

$$d(g^{n_0}(y), g^{n_0}(z)) > d(g^{n_0}(x), g^{n_0}(z)) + d(g^{n_0}(x), g^{n_0}(y)).$$

Then one can get a contradiction by the triangle inequality. □

**Definition 3.11.** *For a compact metric space  $X$  and a homeomorphism  $T : X \rightarrow X$ ,  $T$  is asymptotically entropy expansive if for any  $\varepsilon > 0$ , there is  $\alpha > 0$  such that for any  $x \in X$ , we have*

$$h(B_{\infty}(x, \alpha)) < \varepsilon.$$

We have the following corollary directly:

**Corollary 3.12.** *Assume that a compact invariant set  $\Lambda$  admits a dominated splitting  $T_{\Lambda}M = E \oplus F$  without mixed behavior. Then  $f|_{\Lambda}$  is asymptotically entropy expansive.*

Thus, we also have a “principal symbolic extension”.

**Definition 3.13.** *We say a compact invariant set  $\Lambda$  admits a principal symbolic extension if there is  $n \in \mathbb{N}$  and a compact invariant subset  $\Sigma$  of the shift  $(\{1, 2, \dots, n\}^{\mathbb{Z}}, \sigma)$ , where  $\sigma$  is the shift map, and a continuous surjective map  $\pi : \Sigma \rightarrow \Lambda$  such that for any invariant measure  $\mu$  of  $(\Sigma, \sigma)$ , the metric entropy of  $\mu$  w.r.t.  $\sigma$  is the same as the metric entropy of  $\pi_*(\mu)$  w.r.t.  $f$ .*

It was proven by [3] that any asymptotically entropy expansive system admits a principal symbolic extension. Hence we have the following corollary directly:

**Corollary 3.14.** *Assume that a compact invariant set  $\Lambda$  admits a dominated splitting  $T_{\Lambda}M = E \oplus F$  without mixed behavior. Then  $\Lambda$  admits a principal symbolic extension.*

### 3.6 Upper semi continuous of the metric entropy

Now we can give the proof of Theorem 3.1.

*Proof of Theorem 3.1.* Given a regular partition  $\mathcal{B}$  of  $\mu$ , for any  $\varepsilon > 0$ , there is  $n \in \mathbb{N}$ , for any  $m \in \mathbb{N}$  large enough, by using Lemma 2.2, we have

$$\begin{aligned} h_\mu(f) &\geq h_\mu(f, \mathcal{B}) \\ &\geq \frac{1}{n} H_\mu\left(\bigvee_{i=0}^{n-1} f^{-i}(\mathcal{B})\right) - \varepsilon \\ &\geq -2\varepsilon + \frac{1}{n} H_{\mu_m}\left(\bigvee_{i=0}^{n-1} f_m^{-i}(\mathcal{B})\right) \\ &\geq -2\varepsilon + h_{\mu_m}(f_m, \mathcal{B}). \end{aligned}$$

By [1, Theorem 3.5], we have that for any partition  $\mathcal{B}$  whose norm is less than  $\alpha$ , we have

$$h_{\mu_m}(f_m|_{\Lambda_{f_m}}) \leq h_{\mu_m}(f, \mathcal{B}) + h_\alpha(f_m|_{\Lambda_{f_m}}).$$

By applying Lemma 3.10, one can choose  $\alpha > 0$  such that  $h_\alpha(f_m|_{\Lambda_{f_m}}) < \varepsilon$  for  $m$  large enough. Hence, by taking an  $\alpha$ -regular partition  $\mathcal{B}$ , we have

$$\begin{aligned} h_\mu(f) &\geq -2\varepsilon + h_{\mu_m}(f_m, \mathcal{B}) \geq h_{\mu_m}(f_m|_{\Lambda_{f_m}}) - h_\alpha(f_m|_{\Lambda_{f_m}}) - 2\varepsilon \\ &\geq h_{\mu_m}(f_m) - 3\varepsilon \end{aligned}$$

for  $m$  large enough. By taking a limit and by the arbitrariness of  $\varepsilon$ , one can get the conclusion.  $\square$

## 4 The equilibrium state of $\psi(x) = -\log |\det Df|_{F(x)}|$

In this section, we will consider a  $C^1$  diffeomorphism  $f$  that has a topological attractor with a dominated splitting  $T_\Lambda M = E \oplus F$ . We can extend the bundles  $E$  and  $F$  into a small neighborhood  $U$  of  $\Lambda$  continuously. The extensions are still denoted by  $E$  and  $F$ . We can also extend the function  $\psi(x) = -\log |\det Df|_{F(x)}|$  in a small neighborhood of  $\Lambda$ . In  $U$ , one can define the cone field  $\mathcal{C}_\theta^F$  associated to  $F$  of width  $\theta > 0$  by the following way:

$$\mathcal{C}_\theta^F(x) = \{v = v^E + v^F \in T_x M : |v^E| \leq \theta |v^F|\}.$$

Since the splitting is dominated, the cone field  $\mathcal{C}_\theta^F$  is positive invariant for some large iteration  $Df^N$  and the width of  $Df^n(\mathcal{C}_\theta^F(x))$  tends to zero exponentially for some  $x \in U$  by some uniform constants.

For the continuous function  $\psi = -\log |\det Df|_F|$  and  $n \in \mathbb{N}$ , define

$$S_n \psi(x) = \sum_{i=0}^{n-1} \psi(f^i(x)).$$

Some similar version of the following theorem has been already stated in [9]. The proof based on volume estimation used in [14], originally from [2].

**Theorem 4.1.** *Let  $\Lambda$  be a topological attractor which admits a dominated splitting  $T_\Lambda M = E \oplus F$ . Assume that the entropy function is upper semi continuous, then there is  $\delta_0 > 0$  and  $\theta > 0$  such that for any manifold  $D$  tangent to the cone  $\mathcal{C}_\theta^F$ , whose diameter is less than  $\delta_0$ , then for Lebesgue almost every point  $x \in D$ , for any accumulation point  $\mu$  of*

$$\left\{ \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x)} \right\},$$

*we have*

$$h_\mu(f) + \int \psi d\mu \geq 0.$$

By the properties of cone fields, there are  $\theta > 0$  and  $r > 0$  such that for any disc  $D$  tangent to the cone field  $\mathcal{C}_\theta^F$  and whose diameter is less than  $r$ , if the diameter of  $f^n(D)$  is also less than  $r$ , then  $f^n(D)$  tangent to the cone  $\mathcal{C}_\theta^F$ .

For  $\varepsilon > 0$ , we consider the set of invariant measures:

$$\mathcal{M}_\varepsilon = \{ \mu : h_\mu(f) + \int \psi d\mu \geq -\varepsilon. \}$$

Notice that we have

$$\mathcal{M}_0 = \bigcap_{n \in \mathbb{N}} \mathcal{M}_{1/n}.$$

By the upper semi continuity of the metric entropy, we have that  $\mathcal{M}_\varepsilon$  is closed. Thus  $\mathcal{M} \setminus \mathcal{M}_\varepsilon$  is open. Thus in the metric space of invariant measures, there are countably many open sets  $\{\mathcal{O}_i\}_{i \in \mathbb{N}}$  such that

- the union of all  $\mathcal{O}_i$  is  $\mathcal{M} \setminus \mathcal{M}_\varepsilon$ .
- Each  $\mathcal{O}_i$  is convex and open.
- the closure of  $\mathcal{O}_i$  is contained in  $\mathcal{M} \setminus \mathcal{M}_\varepsilon$ .

For each set  $\mathcal{O}$ , we define

$$B_D(\mathcal{O}) = \{ x \in D : \left\{ \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x)} \right\} \text{ has an accumulation point in } \mathcal{O} \}.$$

$$B_D(\mathcal{O}, n) = \{ x \in D : \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x)} \in \mathcal{O} \}$$

From the definition, we have

$$B_D(\mathcal{O}) \subset \limsup_{n \rightarrow \infty} B_n(\mathcal{O}, n) = \bigcap_{n \geq 1} \bigcup_{m \geq n} B_D(\mathcal{O}, m)$$

We have the following result to conclude Theorem 4.1.

**Lemma 4.2.** *For each  $\mathcal{O} \in \{\mathcal{O}_i\}$ , we have that the Lebesgue measure of  $B_D(\mathcal{O})$  is zero.*



*Proof of Theorem 4.1.* By Lemma 4.2, for any small  $C^1$  sub manifold  $D$  tangent to the cone field  $\mathcal{C}_\theta^F$ , we have that Lebesgue almost every point  $x$  in  $D$ , any accumulation point  $\mu$  of  $\{\frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x)}\}$ , we have that  $h_\mu(f) + \int \psi d\mu \geq 0$ . A small neighborhood of  $\Lambda$  can be foliated by such kind of sub manifolds. Thus, the proof can be complete.  $\square$

Now we give the proof of Lemma 4.2.

*Proof of Lemma 4.2.* By using the Borel-Cantelli argument, for proving  $\text{Leb}(B_D(\mathcal{O})) = 0$ , it suffices to prove that

$$\sum_{n=1}^{\infty} \text{Leb}(B_D(\mathcal{O}, n)) < \infty.$$

Thus we need to estimate  $\text{Leb}(B_D(\mathcal{O}, n))$  for  $n$  large enough.

We consider

$$B_D(\overline{\mathcal{O}}, n) = \{x \in D : \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x)} \in \overline{\mathcal{O}}\}$$

We first cover  $B_D(\overline{\mathcal{O}}, n)$  by a maximal  $(n, \delta)$ -separated set  $\Delta_{n, \delta}$ . Since it is maximal, we have

$$B_D(\mathcal{O}, n) \subset B_D(\overline{\mathcal{O}}, n) \subset \bigcup_{x \in \Delta_{n, \delta}} B_n(x, \delta).$$

We need to choose two constants. Notice that by positive iterations, the cone  $\mathcal{C}_\theta^F$  will decrease exponentially. Thus, by considering a positive iteration of  $D$  (saying  $f^N(D)$ ) and then dividing the positive iteration into small pieces, one can assume that  $D$  is tangent to a very thin cone field (since  $f^N(D)$  is tangent to a very thin cone field).

We can choose constants  $C_\delta$  such that for any disc  $W$  tangent to the cone field  $\mathcal{C}_\theta^F$ , for any points  $x, y \in W$  satisfying  $d_W(x, y) < \delta$ , we have  $|\psi(x) - \psi(y)| \leq \log C_\delta$ . By the uniform continuity of  $\psi$ , one can assume that  $C_\delta \rightarrow 1$  as  $\delta \rightarrow 0$ .

For any  $\kappa > 0$ , there is  $\theta_\kappa$  such that for any disc  $W$  tangent to the cone field  $\mathcal{C}_{\theta_\kappa}^F$ , we have for any  $x \in W$ ,

$$|\log |\det Df|_{T_x W}| - \log \psi(x)| < \kappa.$$

There is  $N_\kappa \in \mathbb{N}$  such that for any  $n > N_\kappa$ , for any sub-manifold  $W$  tangent to  $\mathcal{C}_\theta^F$ , then  $f^n(W)$  is tangent to  $\mathcal{C}_{\theta_\kappa}^F$ .

Thus, there is  $C_\kappa$  (large) such that

$$\begin{aligned} \text{Leb}(B_D(\overline{\mathcal{O}}, n)) &\leq \sum_{x \in \Delta_{n, \delta}} \text{Leb} B_n(x, \delta) = \sum_{x \in \Delta_{n, \delta}} \int_{B_n(x, \delta)} d\text{Leb}_D(y) \\ &= \sum_{x \in \Delta_{n, \delta}} \int_{f^n(B_n(x, \delta))} \prod_{i=0}^n |\det(Df|_{T_{f^{-n+i(z)}f^i W}})|^{-1} d\text{Leb}_{f^n D}(z) \\ &\leq C_\kappa e^{n\kappa} \sum_{x \in \Delta_{n, \delta}} \int_{f^n(B_n(x, \delta))} e^{S_n \psi(z)} d\text{Leb}_{f^n D}(z) \\ &\leq V_\delta C_\kappa e^{n\kappa} C_\delta^n \sum_{x \in \Delta_{n, \delta}} e^{S_n \psi(x)}, \end{aligned}$$

where  $V_\delta$  is the maximal volume of a disc  $D$  whose diameter is less than  $\delta$ , which is tangent to  $\mathcal{C}_\theta^F$ .

Now we need to estimate  $\sum_{x \in \Delta_{n,\varepsilon}} e^{S_n \psi(x)}$ . Take  $\nu_n$  and  $\mu_n$ :

$$\nu_n = \frac{\sum_{x \in \Delta_{n,\varepsilon}} e^{S_n \psi(x)} \delta_x}{\sum_{x \in \Delta_{n,\varepsilon}} e^{S_n \psi(x)}}$$

$$\mu_n = \frac{1}{n} \sum_{i=0}^{n-1} f_*^i \nu_n = \sum_{x \in \Delta_{n,\varepsilon}} \frac{e^{S_n \psi(x)}}{\sum_{x \in \Delta_{n,\varepsilon}} e^{S_n \psi(x)}} \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x)}.$$

**Claim.**  $\mu_n \in \overline{\mathcal{O}}$ .

*Proof of the Claim.* Since  $x \in \Delta_{n,\varepsilon} \subset B_D(\overline{\mathcal{O}}, n)$ , we have that

$$\frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x)} \in \overline{\mathcal{O}}.$$

By the convexity of  $\overline{\mathcal{O}}$ , the claim is true. □

We have that any accumulation point  $\mu$  of  $\{\mu_n\}$  is invariant. And moreover  $\mu \in \overline{\mathcal{O}}$ . By the construction of  $\overline{\mathcal{O}}$ , we have  $h_\mu(f) + \int \psi d\mu \leq -\varepsilon$ .

Now we want to prove

$$h_\mu(f) + \int \psi d\mu \geq \limsup_{n \rightarrow \infty} \frac{1}{n} \log \sum_{x \in \Delta_{n,\delta}} e^{S_n \psi(x)}.$$

Take a partition  $\mathcal{B} = \{B_1, B_2, \dots, B_k\}$  that is  $\delta$ -regular for  $\mu$ . Then we have that every element of  $\bigvee_{i=0}^{n-1} f^{-i} \mathcal{B}$  contains at most one point in  $\Delta_{n,\delta}$ .

By [21, Chapter 9], we have

$$H_{\nu_n}(\bigvee_{i=0}^{n-1} f^{-i} \mathcal{B}) + \int S_n \psi d\nu_n = \log \sum_{x \in \Delta_{n,\delta}} e^{S_n \psi(x)}.$$

Now we need to consider the relationship between  $H_{\mu_n}$  and  $H_{\nu_n}$ .

Given some integer  $1 \leq j < q < n$ , the partition  $\bigvee_{i=0}^{n-1} f^{-i} \mathcal{B}$  can be written in the following way:

$$\bigvee_{i=0}^{n-1} f^{-i} \mathcal{B} = \bigvee_{r=0}^{[(n-j)/q]-1} f^{-(rq+j)} \bigvee_{i=0}^{q-1} f^{-i} \mathcal{B} \vee \bigvee_{\ell \in S_j} f^{-\ell} \mathcal{B},$$

where  $S_j = \{0, 1, \dots, j-1, j + [(n-j)/q]q, \dots, n-1\}$ . We have  $|S_j| \leq 2q$ , and

$$\begin{aligned}
\log \sum_{x \in \Delta_{n,\delta}} e^{S_n \psi(x)} &= H_{\nu_n} \left( \bigvee_{j=0}^{n-1} f^{-j} \mathcal{B} \right) + \int S_n \psi d\nu_n \\
&\leq \sum_{r=0}^{[(n-j)/q]-1} H_{\nu_n} (f^{-(rq+j)} \bigvee_{i=0}^{q-1} f^{-i} \mathcal{B}) + H_{\nu_n} \left( \bigvee_{k \in S_j} f^{-k} (\mathcal{B}) \right) + \int S_n \psi d\nu_n \\
&\leq \sum_{r=0}^{[(n-j)/q]-1} H_{f_*^{rq+j} \nu_n} \left( \bigvee_{i=0}^{q-1} f^{-i} \mathcal{B} \right) + 2q \log k + \int S_n \psi d\nu_n
\end{aligned}$$

By taking the sum for  $j$  from 0 to  $q-1$ , and using Lemma 2.3, we have

$$\begin{aligned}
q \log \sum_{x \in \Delta_{n,\delta}} e^{S_n \psi(x)} &\leq \sum_{p=j}^{j+[(n-j)/q]q} H_{f_*^p \nu_n} \left( \bigvee_{i=0}^{q-1} f^{-i} \mathcal{B} \right) + 2q^2 \log k + q \int S_n \psi d\nu_n \\
&\leq \sum_{p=0}^{n-1} H_{f_*^p \nu_n} \left( \bigvee_{i=0}^{q-1} f^{-i} \mathcal{B} \right) + 2q^2 \log k + q \int S_n \psi d\nu_n \\
&\leq n H_{\mu_n} \left( \bigvee_{i=0}^{q-1} f^{-i} \mathcal{B} \right) + 2q^2 \log k + q \int S_n \psi d\nu_n.
\end{aligned}$$

By dividing by  $n$ , we have

$$\frac{q}{n} \log \sum_{x \in \Delta_{n,\varepsilon}} e^{S_n \psi(x)} \leq H_{\mu_n} \left( \bigvee_{i=0}^{q-1} f^{-i} \mathcal{B} \right) + \frac{2q^2 \log k}{n} + q \int \psi d\mu_n.$$

By taking the lim sup for  $n$  we have

$$q \limsup_{n \rightarrow \infty} \frac{1}{n} \log \sum_{x \in \Delta_{n,\delta}} e^{S_n \psi(x)} \leq H_{\mu} \left( \bigvee_{i=0}^{q-1} f^{-i} \mathcal{B} \right) + q \int \psi d\mu.$$

By dividing  $q$  and letting  $q \rightarrow \infty$ , we have

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \sum_{x \in \Delta_{n,\delta}} e^{S_n \psi(x)} \leq h_{\mu}(\mathcal{B}) + \int \psi d\mu.$$

Recall that  $h_{\mu}(f) + \int \psi d\mu \leq -\varepsilon$ , we have

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \sum_{x \in \Delta_{n,\delta}} e^{S_n \psi(x)} \leq -\varepsilon.$$

Thus,

$$\begin{aligned}
\limsup_{n \rightarrow \infty} \frac{1}{n} \log \text{Leb}(B_D(\mathcal{O}, n)) &\leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log V_{\delta} C_{\kappa} + \kappa + \log C_{\delta} \\
&\quad + \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{x \in \Delta_{n,\delta}} e^{S_n \psi(x)} \\
&\leq \kappa + \log C_{\delta} - \varepsilon.
\end{aligned}$$

By choosing  $\kappa > 0$  small and  $C_\delta$  close to 1, we have that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \text{Leb}(B_D(\mathcal{O}, n)) < 0.$$

Then by using the Borel-Cantelli argument, we can complete the proof.  $\square$

*Proof of the main theorem.* Now we assume that  $\Lambda$  is a topological attractor that admits a dominated splitting  $T_\Lambda M = E \oplus F$  without mixed behavior. Notice that the entropy function is upper semi continuous by Corollary 3.2. Then by Theorem 4.1 we have that there is a measure  $\mu$  such that

$$h_\mu(f) \geq \int \log |\text{Det} Df|_F d\mu.$$

Since there is no mixed behavior, we have that

$$h_\mu(f) \geq \int \sum \lambda_+ d\mu,$$

Where  $\sum \lambda_+$  is the sum of positive Lyapunov exponents of  $\mu$ . On the other hand, by Ruelle's inequality, we have

$$h_\mu(f) \leq \int \sum \lambda_+ d\mu.$$

Thus  $\mu$  satisfies the entropy formula.  $\square$

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Dawei Yang

School of Mathematical Sciences

Soochow University, Suzhou, 215006, P.R. China

yangdw1981@gmail.com, yangdw@suda.edu.cn

Yongluo Cao

School of Mathematical Sciences

Soochow University, Suzhou, 215006, P.R. China

ylcao@suda.edu.cn